University of Rochester

ME 481 – Mechanical Behavior of Materials

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Term paper

Discussion on Ogden's Model for Incompressible Hyperelastic Materials

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INTRODUCTION

Throughout the years, Mechanical Engineers have been developing and refining mathematical models to describe how various materials would react in different situations, a quest that Ogden continued in this paper. Ogden was interested in the deformation of incompressible, isotropic, highly elastic materials with negligible hysteresis effects and negligible temperature changes that follow a strain energy density function. More colloquially Ogden was investigating structural analysis of incompressible hyperelastic materials or incompressible Green-elastic materials, things like rubbers and some biological tissue.

Due to the highly elastic nature of these materials, it is required for the models to be valid at very large displacements. Hence, finite strain theory is used to describe these materials. Within this theory, the tensors used to describe strain differ depending on whether Lagrangian and Eulerian reference frames are used. This discrepancy invites the use of principal stretches to be used to describe the deformation state of a body, instead of strain which works well for infinitesimal strain theory. The stretch tensors (and rotation tensor) are the components produced by the polar decomposition of the deformation gradient; critically, the principal values of both stretch tensors are identical and are therefore used for material models.

While Ogden did not spend time talking about why stretch tensors were used, he did explicitly outline why the eigenvectors of the stretch tensors were the same as the Cauchy stress tensor. Thus, the principal stresses are a function of the principal stretches. The principal stress is specifically defined the following way: $\sigma_i = a_i \frac{\partial \phi}{\partial a_i} - p$ where σ_i is a principal Cauchy stress, a_i is a principal stretch, ϕ is the strain energy density function, and p is an arbitrary hydrostatic stress required for incompressibility. The various material models describing hyperelastic materials all follow this same generic constitutive equation. Furthermore, the only variable that is not dependent on the loading situation is the strain energy density, so the material models differ from each other by defining it in different ways.

One of the first models, the incompressible Neo-Hookean model, used thermomechanical calculations to define the strain energy density as $\phi = \frac{1}{2}\mu(I_1 - 3)$, and while this equation will be reviewed in more depth later in the paper, it serves as a great example to describe the anatomy of these functions. Firstly, note the I_1, this is the first invariant of the stretch tensor, equivalent to $a_1^2 + a_2^2 + a_3^2$ and describes the degree of deformation. Some other models incorporate the second invariant I_2 which is equivalent to $a_1^2 a_2^2 + a_2^2 a_3^2 + a_1^2 a_3^2$. The last invariant is I_3, is sometimes also called J, is defined as $a_1^2 a_2^2 a_3^2$ and describes the compressibility. Of note: all invariants are hydrostatic pressure dependent, so in compressible models the first two invariants are augmented with the third invariant to become pressure independent so that models can keep a similar form when considering compressible materials. Furthermore, it is of note that the stretch of an undeformed body is 1, so the undeformed values of first two invariants are 3 and the third is 1. This explains the deduction from the first invariant from the Neo-Hookean model. Lastly, there are material properties, such as the μ in the Neo-Hookean model that are used to “tune” the model to the physical material.

Ogden actually believed that using invariants was a hinderance and was one of the things he aimed to improve upon. Within the paper, he outlined his own model based on the critique of other models, fit it to data, used it in a practice problem, and determined a relation that ensures realistic results.

RESULTS and CONCLUSION

The ultimate result of this paper is a description of strain energy constructed from a linear combination of 2 variables μ_r and α_r. Using these two Ogden was able to successfully model isotropic elastic materials in several configurations without the use of invariants, an achievement which is still referenced frequently for more modern applications. This hyperelastic model also requires that the material experience no permanent crystallization at high strains meaning the stress strain curve undergoes no hysteresis at extensions well over double the initial length.

The form of Ogden's strain energy is a simple linear combination of two terms $\phi = \mu_r \Psi(\alpha_r)$ where $\Psi(\alpha_r) = \frac{(a_1^{\alpha_r} + a_2^{\alpha_r} + a_2^{\alpha_r} - 3)}{\alpha_r}$. The model was constructed to be applicable to a wide range of isotropic hyperelastic materials and was demonstrated against a common data set collected by Trealor et al to prove its efficacy. This data set was used as an objective measure by which Ogden compared his equation with other elasticity models such as the Varga and Neo-Hookean description and equibiaxial tension. One novel behavior of this model is that it can achieve a high degree of accuracy without the need for an infinite sum. Instead, a discrete sum over a finite set of real numbers leads to a comparatively small fit error even with very few terms. This was one of Ogden's key objectives in the formation of the equation, not just its ability to fit empirical data but to do so while being mathematically simple. Other models such as Alexander or Hart-Smith's models can describe similar material quite accurately but are mathematically cumbersome and are difficult or impossible to use in practical situations.

Ogden goes to great lengths to fit each of Treloar's data sets and in doing so, effectively demonstrates the versatility and simplicity of his model. The first data set records the force vs elongation for a hyper elastic material in simple tension, pure shear, and equiaxial tension. Neo-Hookean and Varga models are used as a comparison for the quality of the curve fit. All graphs show decent agreement at low strains but diverge significantly at elongations over 200%. Using only a single strain energy function $\phi = \mu_1 \Psi(\alpha_1)$ Ogden was able to achieve good fits up to elongations of 200% for each of Treloar's data sets.

The data sets used extended well beyond 200% elongation, which neither the Varga nor Neo-Hookean model successfully replicated for all orientations. Ogden's single term fit was insufficient for such extreme stretches. By adding a second term $\phi = \mu_1\Psi(\alpha_1) + \mu_2\Psi(\alpha_2)$ and scaling the $\frac{\mu_1}{\mu_2}$ ratio such that the second term has negligible impact at stretches below 200%, he was able to retain accuracy at low stretches while also achieving a good fit at more extreme stretches. In particular the simple tension data set was modeled successfully up to stretches of 700%. Unfortunately, the two-term solution was insufficient to model the biaxial tension case for stretches over 200%. Again, a new term was added and the values of μ were adjusted to assure that each term influences a different region of the data set more strongly. Adding the 3rd term not only improves the accuracy of the biaxial tension model but also improves the simple tension and pure sheer models. This implies that additional terms can be added as the need arises to model materials at more extreme stretches.

As a final justification for his equation Ogden checks the restrictions which ensure his model provides a physically reasonable response. To do this Ogden proposes Hill's inequality as the gold standard due to its function for both incompressible but also compressible solids. By solving the Hill equality using Ogden's strain energy he was able to show that the main constraint on his equation is that the value of $\mu_r\,\alpha_r$ must be non-zero and positive. This ensures that the strain energy equation is both positive and definite, which are necessary criteria to mimic material behaviors. The solution to Hill's inequality could not provide any relevant bounds for the individual values of μ_r or α_r only that they must be non-zero with the same sign. This intuitive relation to physical phenomena was the final feather in the cap of a truly robust and impactful model.

ORIGINAL CONTRIBUTION OF OGDEN'S MODEL

In proposing a new strain-energy density function of hyperelastic materials, Ogden also discussed the strong points and shortcomings of previous proposed models, such as the Treloar's model (neo-Hookean form) and the Mooney-Rivlin one.

To begin with, Treloar called his model neo-Hookean because it was developed based on Hooke's law [2].

$$\phi = \frac{1}{2}\mu(I_1 - 3) \tag{1}$$

With μ is the shear modulus and I_1 is the first stretch invariant explained in Introduction. Following the neo-Hookean form, Mooney and Rivlin further developed Treloar's model from depending linearly only on the first invariant I_1 to depending linearly on both the first and second invariants I_1 and I_2 [3].

$$\phi = C_{10}(I_1 - 3) + C_{01}(I_2 - 3) \tag{2}$$

with C_{10} and C_{01} relate to the materials properties. Or in the general form,

$$\phi = \sum_{m,n} C_{mn}(I_1 - 3)^m(I_2 - 3)^n \tag{3}$$

While Treloar and Mooney-Rivlin developed their strain energy density function using stretch invariants as independent variables, in this paper, Ogden expressed his function in terms of the principal stretch ratios.

$$\phi = \sum_r \frac{\mu_r}{\alpha_r}(a_1{}^{\alpha_r} + a_2{}^{\alpha_r} + a_3{}^{\alpha_r} - 3) \tag{4}$$

Using principal stretch ratios $a_{1,2,3}$ instead of stretch invariants $I_{1,2}$ provides the advantage of directly applying the test data into the function, meaning it is mathematically simple. The neo-Hookean function (Equation 1) is similarly simple as the Ogden's model, but the range of validity is different and will be discussed later in this section. For the Mooney-Rivlin's model, while Equation 2 represents the strain energy density function within linearity, the generalized form (Equation 3) is used to guarantee the accuracy once the non-linearity strain range is reached. Even though Equation 3 satisfies the accuracy of the calculation, with $I_1 = a_1{}^2 + a_2{}^2 + a_3{}^2$ and $I_2 = a_1{}^{-2} + a_2{}^{-2} +$

$a_3{}^{-2}$, the access to the principal stretches $a_{1,2,3}$ within the function is being restricted, and therefore the derivation of the function is cumbersome.

It was reported by Ogden that the neo-Hookean model (Equation 1) is valid up to only about 200% strain in simple tension, pure shear, and equibiaxial tests. Additionally, as reported by WESLIM, this model only applies up to 30 – 40% of strain in uniaxial tension [4]. Similarly, The Mooney-Rivlin's model (Equation 2) does not fit Treloar's experimental data very well, and therefore limits its strain range at only up to 150% strain [5]. In comparison, Ogden's strain energy density function (Equation 3) can represent Treloar's experimental data greatly up to 750% strain by adding (α_r, μ_r) terms and gradually adjust those constants. Hence, Ogden's model is applicable for both small and large strain ranges, while the neo-Hookean and Mooney-Rivlin's functions can only correctly represent the mechanical behavior of the material up to 200% strain. It is also reported by Kim. B. et al that for Chloroprene rubber, another rubberlike solid, the neo-Hookean and Mooney-Rivlin models can only be applied at small strain ranges, while the Ogden function covers a larger range of strain of up to 700% strain [6,7].

Additionally, the method of adding and gradually adjusting (α_r, μ_r) terms for Ogden's model is greatly approachable and applicable to any other rubberlike solids due to the linearity of Ogden's function.

CRITICAL APPRECIATION

The assumptions made within the paper are the assumptions used to describe the material and the kind of analysis this model can be used for. The material is assumed to be highly elastic, which means that any time dependent behavior is negligible (no viscous heating) and large displacements are permitted. The completely elastic nature of the material is slightly idealized, however tiny temperature effects are negligible for structural analysis. Although it is typical for hyperelastic materials to exhibit crystallization proportional to strain, it is shown to have a negligible effect on the symmetry of material properties. Furthermore, because the crystals disappear as the strain is removed, the assumption of hysteresis effects being negligible is valid in most cases. Lastly, it is common for highly elastic materials to have large bulk moduli, so incompressibility is also widely applicable. So far, the material described is called an isothermal and incompressible Cauchy-elastic solid, however, Ogden also assumes that stress can be derived from a single strain energy density function or elastic potential function. This is a special case of Cauchy-elastic materials, and the conservative nature of a highly elastic material aligns with this assumption. Overall, the assumptions do not detract from the validity of the model, while it does limit the number of materials that can be described by the model.

The model was intended to be mathematically simple with high accuracy. Considering the fitting to the Treloar dataset, both goals seem to have been accomplished. To speak briefly on any bias within the Treloar dataset, this dataset spans a very large range of strain values over three different modes of deformation that has been widely used as a test dataset for material models beforehand. That being said, multiple datasets could have been used to ensure widespread applicability instead of allowing for the possibility of coincidence or exception.

As for thoroughness, the inclusion of an example situation in Section 5 of the paper where the model was used was excellent. It showed the kinds of situation that this contribution could be used for and showed how to actually work with the new model. The continuity inequality implications discussed at Section 6 of the paper was also a very thorough, as it commented about the restriction on the model to actually convey realistic interactions.

Finally, over 3500 citations have been recorded by google scholar for this paper, with the most recent citation coming from April 15th, 2022. The scientific community obviously values this contribution and are continuing to use it.

GRADE

Ogden successfully addressed two objectives of his strain-energy density function: 1) accurately represents the mechanical behavior of rubberlike solids, and 2) mathematically simple to handle. The author also thoroughly reviewed and compared previously developed models to take out the strong points and address the shortcomings. Additionally, an example solved based on his proposed model and a discussion of the model's limitation using the continuity inequality both show the thoughtfulness of Ogden in validating his model. Therefore, we assign an A+ to this paper.

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